Gerbes of chiral differential operators. III

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Introduction

This note is a sequel to [GII]. Its aim is to "switch on an exterior vector bundle" into the framework of *op. cit.*

Let X be a smooth scheme over a fixed ground ring k containing $1/2$ and E be a vector bundle (i.e. a locally free $\mathcal{O}_X$-module) of finite rank over X. Consider the exterior algebra $\Lambda E = \oplus_{i=0}^{rk(E)} \Lambda^i E$ (over $\mathcal{O}_X$); this is a sheaf of commutative superalgebras over X, where by definition $\mathcal{O}_X$ is purely even and the parity of a component $\Lambda^i E$ is equal to the parity of i.

In this note we study the chiral counterparts of the sheaf $\mathcal{D}_{\Lambda E}$ of superalgebras of differential operators acting on ΛE. Similarly to *op. cit.*, these *chiral sheaves of differential operators on* ΛE exist locally and are by no means unique; the corresponding categories form a *champ en groupoids* $\mathfrak{D}_{\Lambda E}$ over X, called the *gerbe of chiral differential operators on* ΛE.

Our first main result (see Theorem 5.9) says that the characteristic class $c(\mathfrak{D}_{\Lambda E})$ lies in the second hypercohomology group $H^2(X; \Omega_X^{[2,3\rangle})$ (i.e. in the same group where $c(\mathfrak{D}_X)$ lies) and is equal to

$$c(\mathfrak{D}_{\Lambda E}) = c(\Theta_{X/k}) - c(E) = c(\Omega^1_{X/k}) - c(E) \tag{0.1}$$

where $c(E)$ is the "Atiyah-Chern-Simons" class defined in [GII], 7.6. Here $\Theta_{X/k}$ is the tangent bundle. Recall that $\Omega_X^{[2,3\rangle}$ denotes the length 1 complex of sheaves $\Omega^2_{X/k} \longrightarrow \Omega^{3,closed}_{X/k}$, with $\Omega^2_{X/k}$ living in degree 0.

As usually, we obtain in fact a stronger statement, namely the equality (0.1) "on the level of cocycles". As a corollary of this, we conclude that for $E = \Theta_{X/k}$ or $E = \Omega^1_{X/k}$ our gerbes admit a canonical global section. In other words, there exist canonically defined *the* sheaves of chiral do $\mathcal{D}^{ch}_{\Lambda\Theta_X}$ and $\mathcal{D}^{ch}_{\Omega^\cdot_X}$.

Section 6 is devoted to the study of the last sheaf, which is nothing but (the underlying sheaf of) *chiral de Rham complex* from [MSV]. We obtain the transformation laws of 4 local generators of $N = 2$ supersymmetry Q, J, G and L, see Theorem 6.25. In particular, the component Q_0 of the field $Q(z)$ is a globally defined square zero derivation of $\mathcal{D}^{ch}_{\Omega^\cdot_X}$, which is the *chiral de Rham differential* from *op. cit.*

This completes an alternative construction of the chiral de Rham complex sketched in Section 6 of [MSV]. Its difference from the original construction is that it does not use Wick theorem and the arguments of "formal geometry".

In the last section we show that as a simple consequence of the Poincaré-Birkhoff-Witt theorem for $\mathcal{D}^{ch}_{\Omega_X}$ and the Lefschetz fixed point theorem one gets a "moonshine style" formula, cf. Theorem 7.9.

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§1. Preliminaries

1.1. We keep the assumptions of [GII]. We assume that the ground ring k contains 1/2. For a k-supermodule M, we denote by M^{ev} (resp. M^{odd}) the submodule of even (resp. odd) elements, so that $M = M^{ev} \oplus M^{odd}$. For a homogeneous element $a \in M$, we denote by $p(a) \in \mathbb{Z}/2\mathbb{Z}$ its parity. When we speak about graded k-supermodules $M = \oplus_{i\in I} M_i$ we imply that the I-grading is compatible with the parity, i.e. $M^x = \oplus_i M_i^x$ where $M_i^x = M^x \cap M_i,\ x = ev$ or odd.

Let A be a commutative k-superalgebra. A *Lie superalgebroid* over A is a Lie superalgebra over k equipped with a structure of an A-module, such that the identities [GII] (0.2.1) and (0.2.2) hold true.

1.2. A $\mathbb{Z}_{\geq 0}$-*graded vertex superalgebra* (over k) is a $\mathbb{Z}_{\geq 0}$-graded k-supermodule $V = \oplus_{i\geq 0} V_i$ equipped with a distinguished even vector $\mathbf{1} \in V_0$ (*vacuum vector*) and a family of bilinear operations

$$_{(n)}:\ V \times V \longrightarrow V,\ n \in \mathbb{Z},$$

such that

$$p(a_{(n)}b) = p(a) + p(b);\quad V_{i(n)}V_j \subset V_{i+j-n-1} \tag{1.2.1}$$

The following properties must hold:

$$\mathbf{1}_{(n)}a = \delta_{n,-1}a;\ a_{(n)}\mathbf{1} = 0 \text{ for } n \geq 0,\ a_{(-1)}\mathbf{1} = a \tag{1.2.2}$$

and

$$\sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)}b)_{(m+l-j)}c =$$

$$= \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \left\{ a_{(m+n-j)}b_{(l+j)}c - (-1)^{n+p(a)p(b)} b_{(n+l-j)}a_{(m+j)}c \right\} \tag{1.2.3}$$

for all $m, n, l \in \mathbb{Z}$. A particular case of (1.2.3) corresponding to $m = 0$:

$$(a_{(n)}b)_{(l)}c = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \left\{ a_{(n-j)}b_{(l+j)}c - (-1)^{n+p(a)p(b)} b_{(n+l-j)}a_{(j)}c \right\} \tag{1.2.4}$$

Setting $n = l = -1$ we get

$$(a_{(-1)}b)_{(-1)}c = \sum_{j=0}^{\infty} \left\{ a_{(-1-j)}b_{(-1+j)}c + (-1)^{p(a)p(b)} b_{(-2-j)}a_{(j)}c \right\} \tag{1.2.5}$$

In the sequel we shall work only with $\mathbb{Z}_{\geq 0}$-graded vertex superalgebras, and call them simply vertex superalgebras. This $\mathbb{Z}_{\geq 0}$-grading will be called the grading by *conformal weight*.

1.3. Let V be a vertex superalgebra. The even operators $\partial^{(j)}: V \longrightarrow V$ of degree j ($j \in \mathbb{Z}_{\geq 0}$) are defined in the same manner as in [GII], (0.5.5), and they satisfy [GII], (0.5.7), (0.5.8), (0.5.10) and (0.5.11). The "supercommutativity" formula reads as

$$a_{(n)}b = (-1)^{n+p(a)p(b)+1} \sum_{j \geq 0} (-1)^j \partial^{(j)}(b_{(n+j)}a) \tag{1.3.1}$$

and we have the usual OPE formula

$$[a_{(m)}, b_{(n)}] = \sum_{j \geq 0} \binom{m}{j} (a_{(j)}b)_{(m+n-j)} \tag{1.3.2}$$

where in the left hand side stands the supercommutator

$$[a_{(m)}, b_{(n)}] := a_{(m)}b_{(n)} - (-1)^{p(a)p(b)} b_{(n)}a_{(m)} \tag{1.3.3}$$

§2. Vertex Superalgebroids

2.1. An *extended Lie superalgebroid* (over A) is a quintuple $\mathcal{T} = (A, T, \Omega, \partial, \langle , \rangle)$ where T is a Lie superalgebroid over A, Ω is an A-module equipped with a structure of a module over the Lie superalgebra T, $\partial: A \longrightarrow \Omega$ an even A-derivation and a morphism of T-modules, $\langle , \rangle: T \times \Omega \longrightarrow A$ is an even A-bilinear pairing.

The following identities must be true ($a \in A, \tau, \nu \in T, \omega \in \Omega$):

$$\langle \tau, \partial a \rangle = \tau(a) \tag{2.1.1}$$

$$\tau(a\omega) = \tau(a)\omega + (-1)^{p(\tau)p(a)} a\tau(\omega) \tag{2.1.2}$$

$$(a\tau)(\omega) = a\tau(\omega) + \langle \tau, \omega \rangle \partial a \tag{2.1.3}$$

$$\tau(\langle \nu, \omega \rangle) = \langle [\tau, \nu], \omega \rangle + (-1)^{p(\tau)p(\nu)} \langle \nu, \tau(\omega) \rangle \tag{2.1.4}$$

For example to a Lie superalgebroid T one associates canonically an extended superalgebroid with $\Omega = Hom_A(T, A)$, as in [GII], 1.2.

2.2. De Rham - Chevalley complex. Let $\mathcal{T} = (A, T, \Omega, \dots)$ be an extended Lie A-superalgebroid. Let us define A-modules $\Omega^i = \Omega^i(\mathcal{T})$, $i \in \mathbb{Z}_{\geq 0}$, as follows. Set $\Omega^0 = A$, $\Omega^1 = \Omega$. For $i \geq 2$, Ω^i is the submodule of the module of A-polylinear homomorphisms h from T^{i-1} to Ω such that the function $\langle \tau_1, h(\tau_2, \dots, \tau_i) \rangle$ is skew symmetric (in the graded sense) with respect to all permutations of $(\tau_1, \dots, \tau_i)$.

For example, if $\mathcal{T}$ is associated with a Lie superalgebroid T as above then $\Omega^i = Hom_A(\Lambda^i_A T, A)$.

Let us define the even maps $d_{DR} = d^i_{DR}:\ \Omega^i \longrightarrow \Omega^{i+1}$ as follows. For $i = 0$ we set $d_{DR}a = -\partial a$. For $i \geq 1$ we set

$$d_{DR}h(\tau_1, \dots, \tau_i) = d_{Lie}h(\tau_1, \dots, \tau_i) - (-1)^{p(h)p(\tau_1)}\partial\langle\tau_1, h(\tau_2, \dots, \tau_i)\rangle \qquad (2.2.1)$$

where

$$d_{Lie}h(\tau_1, \dots, \tau_i) = \sum_{j=1}^{i} (-1)^{j+1+p(\tau_j)(p(\tau_1)+\dots+p(\tau_{j-1}))}\tau_j(h(\tau_1, \dots, \widehat{\tau}_j, \dots, \tau_i)) +$$

$$+ \sum_{1\leq j<l\leq i} (-1)^{j+l+p(\tau_j)(p(\tau_1)+\dots+p(\tau_{j-1}))+p(\tau_l)(p(\tau_1)+\dots+\widehat{p(\tau_j)}+\dots+p(\tau_{l-1}))}\times$$

$$\times h([\tau_j, \tau_l], \tau_1, \dots, \widehat{\tau}_j, \dots, \widehat{\tau}_l, \dots, \tau_i) \qquad (2.2.2)$$

For example,

$$d_{DR}\omega(\tau) = (-1)^{p(\omega)p(\tau)}\{\tau(\omega) - \partial\langle\tau, \omega\rangle\}, \qquad (2.2.3)$$

for $\omega \in \Omega^1 = \Omega$; and

$$d_{DR}h(\tau_1, \tau_2) = -h([\tau_1, \tau_2]) + (-1)^{p(h)p(\tau_1)}\tau_1(h(\tau_2)) -$$

$$-(-1)^{p(\tau_2)(p(h)+p(\tau_1))}\tau_2(h(\tau_1)) - (-1)^{p(h)p(\tau_1)}\partial\langle\tau_1, h(\tau_2)\rangle, \qquad (2.2.4)$$

for $h \in \Omega^2$.

Let us introduce the action of the Lie algebra T on the modules Ω^i by

$$\tau(h)(\tau_1, \dots, \tau_{i-1}) = \tau(h(\tau_1, \dots, \tau_{i-1})) -$$

$$-\sum_{j=1}^{i-1} (-1)^{p(\tau)(p(\tau_1)+\dots+p(\tau_{j-1}))}h(\tau_1, \dots, [\tau, \tau_j], \dots, \tau_i) \qquad (2.2.5)$$

Let us define the convolution operators $\langle\tau, \cdot\rangle:\ \Omega^i \longrightarrow \Omega^{i-1}$ by

$$\langle\tau, h\rangle(\tau_1, \dots, \tau_{i-2}) = (-1)^{p(\tau)p(h)}h(\tau, \tau_1, \dots, \tau_{i-2}) \qquad (2.2.6)$$

The maps $\{d^i_{DR}\}$ may be characterized as a unique collection of maps such that $d^0_{DR} = -\partial$ and the *Cartan formula*

$$\tau(h) = \langle\tau, d_{DR}h\rangle + d_{DR}\langle\tau, h\rangle \qquad (2.2.7)$$

holds true.

The maps d_{DR} commute with the action of T. One checks that $d^2_{DR} = 0$, so we get a complex $(\Omega^\bullet(\mathcal{T}), d_{DR})$ called the *de Rham-Chevalley complex* of $\mathcal{T}$.

2.3. A *vertex superalgebroid* is a septuple $\mathcal{A} = (A, T, \Omega, \partial, \gamma, \langle\ ,\ \rangle, c)$ where A is a supercommutative k-algebra, T is a Lie superalgebroid over A, Ω is an A-module equipped with an action of the Lie superalgebra T, $\partial:\ A \longrightarrow \Omega$ is an even derivation commuting with the T-action,

$$\langle\ ,\ \rangle:\ (T \oplus \Omega) \times (T \oplus \Omega) \longrightarrow A$$

is a supersymmetric even k-bilinear pairing equal to zero on $\Omega\times\Omega$ and such that $\mathcal{T}_{\mathcal{A}}=(A,T,\Omega,\partial,\langle\ ,\ \rangle|_{T\times\Omega})$ is an extended Lie superalgebroid over A; $c:\ T\times T\longrightarrow\Omega$ is a skew supersymmetric even k-bilinear pairing and $\gamma:\ A\times T\longrightarrow\Omega$ is an even k-bilinear map.

The following axioms must hold ($a,b\in A;\ \tau,\tau_i\in T$):

$$\gamma(a,b\tau)=\gamma(ab,\tau)-a\gamma(b,\tau)-$$

$$-(-1)^{p(\tau)(p(a)+p(b))}\tau(a)\partial b-(-1)^{p(a)p(b)+p(\tau)p(a)+p(\tau)p(b)}\tau(b)\partial a \qquad (A1)$$

$$\langle a\tau_1,\tau_2\rangle=a\langle\tau_1,\tau_2\rangle+\langle\gamma(a,\tau_1),\tau_2\rangle-(-1)^{p(a)(p(\tau_1)+p(\tau_2))}\tau_1\tau_2(a) \qquad (A2)$$

$$c(a\tau_1,\tau_2)=ac(\tau_1,\tau_2)+\gamma(a,[\tau_1,\tau_2])-$$

$$-(-1)^{p(\tau_2)(p(\tau_1)+p(a))}\gamma(\tau_2(a),\tau_1)+(-1)^{p(\tau_2)(p(\tau_1)+p(a))}\tau_2(\gamma(a,\tau_1))-$$

$$-(-1)^{p(a)(p(\tau_1)+p(\tau_2))}\frac{1}{2}\langle\tau_1,\tau_2\rangle\partial a+(-1)^{p(a)(p(\tau_1)+p(\tau_2))}\frac{1}{2}\partial\tau_1\tau_2(a)-$$

$$-(-1)^{p(\tau_2)(p(a)+p(\tau_1))}\frac{1}{2}\partial\langle\tau_2,\gamma(a,\tau_1)\rangle \qquad (A3)$$

$$\langle[\tau_1,\tau_2],\tau_3\rangle+(-1)^{p(\tau_1)p(\tau_2)}\langle\tau_2,[\tau_1,\tau_3]\rangle=\tau_1(\langle\tau_2,\tau_3\rangle)-$$

$$-(-1)^{p(\tau_1)p(\tau_2)}\frac{1}{2}\tau_2(\langle\tau_1,\tau_3\rangle)-(-1)^{p(\tau_3)(p(\tau_1)+p(\tau_2))}\frac{1}{2}\tau_3(\langle\tau_1,\tau_2\rangle)+$$

$$+(-1)^{p(\tau_1)p(\tau_2)}\langle\tau_2,c(\tau_1,\tau_3)\rangle+(-1)^{p(\tau_3)(p(\tau_1)+p(\tau_2))}\langle\tau_3,c(\tau_1,\tau_2)\rangle \qquad (A4)$$

$$d_{Lie}c(\tau_1,\tau_2,\tau_3)=-\frac{1}{2}\partial\big\{\langle[\tau_1,\tau_2],\tau_3\rangle+(-1)^{p(\tau_2)p(\tau_3)}\langle[\tau_1,\tau_3],\tau_2\rangle-$$

$$-(-1)^{p(\tau_1)(p(\tau_2)+p(\tau_3))}\langle[\tau_2,\tau_3],\tau_1\rangle-\tau_1(\langle\tau_2,\tau_3\rangle)+(-1)^{p(\tau_1)p(\tau_2)}\tau_2(\langle\tau_1,\tau_3\rangle)-$$

$$-(-1)^{p(\tau_3)(p(\tau_1)+p(\tau_2))}2\langle\tau_3,c(\tau_1,\tau_2)\rangle\big\} \qquad (A5)$$

where d_{Lie} is defined by (2.2.2).

2.4. All the constructions of [GII] generalize to the $\mathbb{Z}/(2)$-graded case in an obvious manner.

§3. Some formulas

3.1. Let A be a smooth k-algebra of relative dimension n, such that the A-module $T=Der_k(A)$ is free and admits a base $\{\bar\tau_i\}$ consisting of commuting vector fields. Let E be a free A-module of rank m, with a base $\{\phi_\alpha\}$. We shall call the set $\mathfrak{g}=\{\bar\tau_i;\ \phi_\alpha\}\subset A\oplus E$ a *frame* of (A,E).

Consider a commutative A-superalgebra $\Lambda E=\oplus_{i=0}^m\Lambda_A^i(E)$ where the parity of $\Lambda_A^i(E)$ is equal to the parity of i. Each frame $\mathfrak{g}$ as above gives rise to a ΛE-base $\{\tau_i;\ \psi_\alpha\}$ of the Lie superalgebroid $T_{\Lambda E}=Der_k(\Lambda E)$, defined as follows. We extend the fields $\bar\tau_i$ to derivations τ_i of the whole superalgebra ΛE by the rule

$$\tau_i(a)=\bar\tau_i(a);\ \tau_i(\sum\ a_\alpha\phi_\alpha)=\sum\ \bar\tau_i(a_\alpha)\phi_\alpha \qquad (3.1.1)$$

(Note that this extension depends on a choice of a base $\{\phi_\alpha\}$ of the module E.) The fields $\{\tau_i\}$ form a ΛE-base of the even part $T^{ev}_{\Lambda E}$.

We define the odd vector fields $\psi_\alpha \in T^{odd}_{\Lambda E}$ by

$$\psi_\alpha(\sum\ a_\nu\phi_\nu) = a_\alpha;\ \psi_\alpha(a) = 0 \tag{3.1.2}$$

These fields form a ΛE-base of $T^{odd}_{\Lambda E}$.

Let $\{\omega_i;\ \rho_\alpha\}$ be the dual base of the module of 1-superforms $\Omega_{\Lambda E} = Hom^{ev}_{\Lambda E}(T_{\Lambda E}, \Lambda E)$, defined by

$$\langle\tau_i, \omega_j\rangle = \delta_{ij};\ \langle\psi_\alpha, \rho_\beta\rangle = \delta_{\alpha\beta};\ \langle\tau_i, \rho_\alpha\rangle = \langle\psi_\alpha, \omega_i\rangle = 0 \tag{3.1.3}$$

3.2. Let us describe the effect of a change of frame. Let $\mathfrak{g}' = \{\bar{\tau}'_i;\ \phi'_\alpha\}$ be another frame, with $\bar{\tau}'_i = g^{ij}\bar{\tau}_j;\ \phi'_\alpha = A^{\alpha\beta}\phi_\beta,\ g = (g^{ij}) \in GL_n(A),\ A = (A^{\alpha\beta}) \in GL_m(A)$.

The corresponding new bases τ'_i, etc. look as follows.

$$\tau'_i = g^{ip}\tau_p + g^{i\alpha\gamma}\phi_\gamma\psi_\alpha \tag{3.2.1}$$

where

$$g^{i\alpha\gamma} = g^{iq}\tau_q(A^{-1\alpha\mu})A^{\mu\gamma} \tag{3.2.2}$$

Next,

$$\psi'_\alpha = A^{-1\mu\alpha}\psi_\mu \tag{3.2.3}$$

$$\omega'_i = g^{-1pi}\omega_p \tag{3.2.4}$$

$$\rho'_\alpha = \tau_i(A^{\alpha\gamma})\phi_\gamma\omega_i + A^{\alpha\mu}\rho_\mu \tag{3.2.5}$$

Formulas for the inverse transformation:

$$\tau_q = g^{-1qi}\tau'_i + \tau_q(A^{\alpha\gamma})\phi_\gamma\psi'_\alpha \tag{3.2.6}$$

$$\psi_\beta = A^{\alpha\beta}\psi'_\alpha \tag{3.2.7}$$

$$\omega_j = g^{pj}\omega'_p \tag{3.2.8}$$

$$\rho_\beta = A^{-1\beta\alpha}\rho'_\alpha + g^{p\beta\gamma}\phi_\gamma\omega'_p \tag{3.2.9}$$

These formulas show that T is canonically an A-module quotient of $T_{\Lambda E}$ and $\Omega = Hom_A(T, A)$ is canonically an A-submodule of $\Omega_{\Lambda E}$. In fact the whole de Rham complex $\Omega^{\cdot}_A$ is canonically the subcomplex of $\Omega^{\cdot}_{\Lambda E}$.

3.3. Recall that

$$g^{ip}\tau_p(g^{jq}) = g^{jp}\tau_p(g^{iq}) \tag{3.3.1}$$

$$g^{ip}\tau_q\tau_p(g^{jq}) = g^{jq}\tau_p\tau_q(g^{ip}) \tag{3.3.2}$$

and

$$\tau_p(g^{-1qr}) = \tau_q(g^{-1pr}) \tag{3.3.3}$$

see [GII], 5.4.

It is easy to see that

$$tr\{\tau_p(A)\tau_q(A^{-1})\} = tr\{\tau_q(A)\tau_p(A^{-1})\} \tag{3.3.4}$$

Using (3.3.1) and (3.3.4) one sees easily that

$$g^{ip}\tau_p(g^{j\nu\nu}) = g^{jq}\tau_q(g^{i\nu\nu}) \tag{3.3.5}$$

3.4. Let $\mathcal{A} = \mathcal{A}_{\Lambda E;\mathfrak{g}}$ be the vertex superalgebroid corresponding to the frame $\mathfrak{g}$. We have the following identities in $\mathcal{A}$:

γ-formulas

$$\gamma(a, b\tau_i) = -\tau_i(a)\partial b - \tau_i(b)\partial a \tag{3.4.1}$$

$$\gamma(a, b\phi_\nu\psi_\mu) = \delta_{\nu\mu} b\partial a \tag{3.4.2}$$

$$\gamma(a\phi_\beta, b\psi_\mu) = \delta_{\beta\mu} a\partial b \tag{3.4.3}$$

$\langle,\rangle$-formulas

$$\langle a\tau_i, b\tau_j\rangle = -b\tau_i\tau_j(a) - a\tau_j\tau_i(b) - \tau_i(b)\tau_j(a) \tag{3.4.4}$$

$$\langle a\phi_\alpha\psi_\beta, b\tau_i\rangle = \delta_{\alpha,\beta} b\tau_i(a) \tag{3.4.5}$$

$$\langle a\phi_\alpha\psi_\beta, \phi_{\alpha'}\psi_{\beta'}\rangle = ab\delta_{\beta\alpha'}\delta_{\beta'\alpha} \tag{3.4.6}$$

c-formulas

$$c(a\tau_i, b\tau_j) = \frac{1}{2}\{\tau_i(b)\partial\tau_j(a) - \tau_j(a)\partial\tau_i(b)\} + \frac{1}{2}\partial\{b\tau_i\tau_j(a) - a\tau_j\tau_i(b)\} \tag{3.4.7}$$

$$c(a\phi_\alpha\psi_\mu, b\phi_\beta\psi_\nu) = \frac{\delta_{\mu\beta}\delta_{\nu\alpha}}{2}\{a\partial b - b\partial a\} \tag{3.4.8}$$

$$c(a\phi_\alpha\psi_\mu, b\tau_i) = -\frac{\delta_{\mu\alpha}}{2}\partial\{b\tau_i(a)\} \tag{3.4.9}$$

$$c(a\tau_i, b\psi_\alpha) = c(a\phi_\alpha\psi_\mu, b\psi_\nu) = 0 \tag{3.4.10}$$

3.5. Let $\mathfrak{g}'$ be another frame as in 3.2. We have

$$\gamma(a, \tau'_p) = \gamma(a, g^{pq}\tau_q + g^{p\mu\nu}\phi_\nu\psi_\mu) =$$

$$= -\tau_q(a)\partial g^{pq} - \tau_q(g^{pq})\partial a + g^{p\mu\mu}\partial a \tag{3.5.1}$$

$$\gamma(a\phi'_\mu, \psi'_\alpha) = \gamma(aA^{\mu\beta}\phi_\beta, A^{-1\nu\alpha}\psi_\nu) = aA^{\mu\beta}\partial A^{-1\beta\alpha} \tag{3.5.2}$$

Next,

$$\langle\tau'_i, \tau'_j\rangle = \langle g^{ip}\tau_p + g^{i\mu\alpha}\phi_\alpha\psi_\mu, g^{jq}\tau_q + g^{j\nu\beta}\phi_\beta\psi_\nu\rangle =$$

$$= -2g^{ip}\tau_q\tau_p(g^{jq}) - \tau_p(g^{jq})\tau_q(g^{ip}) +$$
$$+2g^{ip}\tau_p(g^{j\nu\nu}) + g^{ip}g^{jq}\tau_p(A^{-1\mu\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma\sigma})A^{\sigma\mu} \tag{3.5.3}$$

and

$$\langle\tau_i', \psi_\alpha'\rangle = \langle\psi_\alpha', \psi_\beta'\rangle = 0 \tag{3.5.4}$$

Finally,

$$c(\tau_i', \tau_j') = \frac{1}{2}\{\tau_p(g^{jq})\partial\tau_q(g^{ip}) - \tau_q(g^{ip})\partial\tau_p(g^{jq})\} + \frac{1}{2}\{g^{i\mu\nu}\partial g^{j\nu\mu} - g^{j\nu\mu}\partial g^{i\mu\nu}\} \tag{3.5.5}$$

and

$$c(\tau_i', \psi_\alpha') = c(\psi_\alpha', \psi_\beta') = 0 \tag{3.5.6}$$

§4. Chern-Simons term

This Section is parallel to [GII], Section 5.

4.1. We keep the setup of the previous section. Let $\mathcal{A} = \mathcal{A}_{\Lambda E;\mathfrak{g}},\ \mathcal{A}' = \mathcal{A}_{\Lambda E;\mathfrak{g}'}$ (resp., $\mathcal{B} = \mathcal{B}_{\Lambda E;\mathfrak{g}},\ \mathcal{B}' = \mathcal{B}_{\Lambda E;\mathfrak{g}'}$) be the vertex superalgebroids (resp. prealgebroids) corresponding to our frames.

As in [GII], 5.5 we have a canonical isomorphism

$$g = g_{\mathfrak{g},\mathfrak{g}'} = (Id_{\Lambda E}, Id_{T_{\Lambda E}}, Id_{\Omega_{\Lambda E}}, h) :\ \mathcal{B}' \overset{\sim}{\longrightarrow} \mathcal{B} \tag{4.1.1}$$

where

$$h = h_{\mathfrak{g},\mathfrak{g}'} :\ T_{\Lambda E} \longrightarrow \Omega_{\Lambda E} \tag{4.1.2}$$

is defined by the condition

$$\langle x', h(y')\rangle = -\frac{1}{2}\langle x', y'\rangle, \quad x', y' \in \{\tau_i'\} \cup \{\psi_\alpha'\} \tag{4.1.3}$$

Using (3.5.3) and (3.5.4) we find the following explicit formulas for h:

$$h(\tau_i') = h^{ij}\omega_j;\ h(\psi_\alpha') = 0 \tag{4.1.4}$$

where $h^{ij} = h^{ij}_\Omega - h^{ij}_E$,

$$h^{ij}_\Omega = \tau_p\tau_j(g^{ip}) + \frac{1}{2}\tau_q(g^{ip})\tau_p(g^{rq})g^{-1jr} \tag{4.1.5}$$

cf. [GII], (5.7.2), and

$$h^{ij}_E = \tau_j(g^{i\nu\nu}) + \frac{1}{2}g^{iq}\tau_j(A^{-1\mu\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma\nu})A^{\nu\mu} \tag{4.1.6}$$

The meaning of the notation h_Ω will become clear below, see §6.

4.2. We have

$$\mathcal{A} = g_* \mathcal{A}' \dot{+} \mathfrak{b} \tag{4.2.1}$$

where the closed 3-form $\mathfrak{b} \in \Omega^{3,cl}_{\Lambda E}$ is defined by

$$\mathfrak{b}(x', y') = c(x', y') - x'(h(y')) + (-1)^{p(x')p(y')} y'(h(x')), \tag{4.2.2}$$

$x', y' \in \{\omega'_i\} \cup \{\psi'_\alpha\}$, cf. [GII], (5.7.3).

It is easy to see that $\psi'_\alpha(h(\tau'_i)) = 0$; on the other hand we know already that $h(\psi'_\alpha) = 0$ and $c(\psi'_\alpha, y') = 0$. It follows that $\mathfrak{b} \in \Omega^{3,cl} \subset \Omega^{3,cl}_{\Lambda E}$.

Next, we have

$$\tau'_i(h(\tau'_j)) = (g^{ip}\tau_p + g^{i\alpha\gamma}\phi_\gamma\psi_\alpha)(h^{jq}\omega_q) =$$

(note that the second summand is zero)

$$= (g^{ip}\tau_p)(h^{jq}_\Omega \omega_q) + (g^{ip}\tau_p)(h^{jq}_E \omega_q) = g^{ip}\tau_p(h^{jq}_\Omega)\omega_q + h^{jp}_\Omega \partial g^{ip} -$$

$$-g^{ip}\tau_p(h^{jq}_E)\omega_q - h^{jp}_E \partial g^{ip} \tag{4.2.3}$$

It follows that $\mathfrak{b} = \mathfrak{b}_\Omega - \mathfrak{b}_E$ where $\mathfrak{b}_\Omega, \mathfrak{b}_E \in \Omega^3$ are given by

$$\mathfrak{b}_\Omega(\tau'_i, \tau'_j) = \frac{1}{2}\{\tau_p(g^{jq})\partial\tau_q(g^{ip}) - \tau_q(g^{ip})\partial\tau_p(g^{jq})\} -$$

$$-g^{ip}\tau_p(h^{jq}_\Omega)\omega_q - h^{jp}_\Omega \partial g^{ip} + g^{jp}\tau_p(h^{iq}_\Omega)\omega_q + h^{ip}_\Omega \partial g^{jp} \tag{4.2.4}$$

and

$$\mathfrak{b}_E(\tau'_i, \tau'_j) = -\frac{1}{2}\{g^{i\mu\nu}\partial g^{j\nu\mu} - g^{j\nu\mu}\partial g^{i\mu\nu}\} -$$

$$-g^{ip}\tau_p(h^{jq}_E)\omega_q + g^{jp}\tau_p(h^{iq}_E)\omega_q - h^{jp}_E \partial g^{ip} + h^{ip}_E \partial g^{jp} \tag{4.2.5}$$

The form $\mathfrak{b}_\Omega$ has already been computed in [GII], Magic Lemma 5.6 and Theorem 6.4 (b), and is equal to

$$\mathfrak{b}_\Omega(\tau'_i, \tau'_j) = -\frac{1}{2} tr\{g^{-1}\tau'_i(g)g^{-1}\tau'_j(g)g^{-1}\tau'_r(g) - g^{-1}\tau'_j(g)g^{-1}\tau'_i(g)g^{-1}\tau'_r(g)\}\omega'_r \tag{4.2.6}$$

cf. *loc.cit.* (5.5.3) and (6.4.2). Note that $\mathfrak{b}_\Omega$ is closed, hence $\mathfrak{b}_E$ is closed.

4.3. Magic Lemma. *We have*

$$\mathfrak{b}_E(\tau'_i, \tau'_j) = -\frac{1}{2} tr\{A^{-1}\tau'_i(A)A^{-1}\tau'_j(A)A^{-1}\tau'_r(A) - A^{-1}\tau'_j(A)A^{-1}\tau'_i(A)A^{-1}\tau'_r(A)\}\omega'_r \tag{4.3.1}$$

Proof. Let us denote the six terms in (4.2.5) by A, A', B, B', C and C'. We have

$$A = -\frac{1}{2} g^{iq}\tau_q(A^{-1\mu\alpha})A^{\alpha\nu}\tau_r\{g^{jp}\tau_p(A^{-1\nu\beta})A^{\beta\mu}\}\omega_r =$$

$$= -\frac{1}{2} g^{iq}\tau_q(A^{-1\mu\alpha})A^{\alpha\nu}\{\tau_r(g^{jp})\tau_p(A^{-1\nu\beta})A^{\beta\mu}\omega_r + g^{jp}\tau_r\tau_p(A^{-1\nu\beta})A^{\beta\mu}\omega_r +$$
$$+ g^{jp}\tau_p(A^{-1\nu\beta})\tau_r(A^{\beta\mu})\omega_r\}$$

Next,

$$B = -\frac{1}{2} g^{ip}\tau_p\{g^{jq}\tau_r(A^{-1\mu\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma\nu})A^{\nu\mu}\}\omega_r - g^{ip}\tau_p\tau_r(g^{j\nu\nu})\omega_r =$$
$$= -\frac{1}{2} g^{ip}\tau_p(g^{jq})\tau_r(A^{-1\mu\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma\nu})A^{\nu\mu}\omega_r - \frac{1}{2} g^{ip}g^{jq}\tau_p\tau_r(A^{-1\mu\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma\nu})A^{\nu\mu}\omega_r -$$
$$-\frac{1}{2} g^{ip}g^{jq}\tau_r(A^{-1\mu\beta})\tau_p(A^{\beta\gamma})\tau_q(A^{-1\gamma\nu})A^{\nu\mu}\omega_r - -\frac{1}{2} g^{ip}g^{jq}\tau_r(A^{-1\mu\beta})A^{\beta\gamma}\tau_p\tau_q(A^{-1\gamma\nu})A^{\nu\mu}\omega_r -$$
$$-\frac{1}{2} g^{ip}g^{jq}\tau_r(A^{-1\mu\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma\nu})\tau_p(A^{\nu\mu})\omega_r - g^{ip}\tau_p\tau_r(g^{j\nu\nu})\omega_r$$

Finally,

$$C = -\frac{1}{2} g^{jq}\tau_p(A^{-1\mu\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma\nu})A^{\nu\mu}\tau_r(g^{ip})\omega_r - \tau_r(g^{ip})\tau_p(g^{j\nu\nu})\omega_r$$

We see that $A1 = -C1', A2 = -B'2, B1 = -B'1$ (by (3.4.6)), $B2 = -A'2, B4 = -B'4$ and $C1 = -A'1$. Next,

$$B6 + C2 = -\tau_r\{g^{ip}\tau_p(g^{j\nu\nu})\},$$

so $B6 + C2 = -B'6 - C'2$ by (3.3.5).

So we are left with three terms: $A3, B3$ and $B5$ and their primed partners. It is easy to see that

$$A3 = B3 = -B5 = -\frac{1}{2} tr\{A^{-1}\tau_i'(A)A^{-1}\tau_j'(A)A^{-1}\tau_r'(A)\}\omega_r',$$

which implies the Lemma. △

§5. Atiyah term

This section is parallel to [GII], Section 6.

5.1. We keep the setup of the previous section. Let us denote by $\mathcal{T}_{\Lambda E} = (\Lambda E, T_{\Lambda E}, \Omega_{\Lambda E}, \partial)$ the extended vertex superalgebroid Lie corresponding to our data.

Let $\mathfrak{g}'' = \{\bar\tau_i'', \phi_\alpha''\}$ be a third frame of (A, E), with $\bar\tau_i'' = g'^{ij}\bar\tau_j', \phi_\alpha'' = A'^{\alpha\beta}\phi_\beta'$. We have the corresponding new bases of $T_{\Lambda E}$ and $\Omega_{\Lambda E}$ given by

$$\tau_i'' = g'^{ip}\tau_p' + g'^{i\alpha\gamma}\phi_\gamma'\psi_\alpha' \tag{5.1.1}$$

where

$$g'^{i\alpha\gamma} = g'^{iq}\tau_q'(A'^{-1\alpha\mu})A'^{\mu\gamma} \tag{5.1.2}$$

$$\psi''_\alpha = A'^{-1\mu\alpha}\psi'_\mu \tag{5.1.3}$$

$$\omega''_i = g'^{-1pi}\omega'_p \tag{5.1.4}$$

$$\rho''_\alpha = \tau'_i(A'^{\alpha\gamma})\phi'_\gamma\omega'_i + A'^{\alpha\mu}\rho'_\mu \tag{5.1.5}$$

5.2. Let $\mathcal{A}'' = \mathcal{A}_{\Lambda E,\mathfrak{g}''}$ (resp. $\mathcal{B}'' = \mathcal{B}_{\Lambda E;\mathfrak{g}''}$) be the vertex superalgebroid (resp. prealgebroid) corresponding to the third frame. We have canonical isomorphisms

$$\mathcal{B}'' \overset{g_{\mathfrak{g}',\mathfrak{g}''}}{\underset{\sim}{\longrightarrow}} \mathcal{B}' \overset{g_{\mathfrak{g},\mathfrak{g}'}}{\underset{\sim}{\longrightarrow}} \mathcal{B}$$

as well as the morphism $g_{\mathfrak{g},\mathfrak{g}''}:\ \mathcal{B}'' \overset{\sim}{\longrightarrow} \mathcal{B}$ over $Id_{\mathcal{T}_{\Lambda E}}$, given by functions $h_{\mathfrak{g},\mathfrak{g}'}, h_{\mathfrak{g}',\mathfrak{g}''}, h_{\mathfrak{g},\mathfrak{g}''}$, and we are aiming to compute the discrepancy

$$\mathfrak{a} = \mathfrak{a}_{\mathfrak{g},\mathfrak{g}',\mathfrak{g}''} := h_{\mathfrak{g},\mathfrak{g}'} + h_{\mathfrak{g}',\mathfrak{g}''} - h_{\mathfrak{g},\mathfrak{g}''} \in \Omega^2_{\Lambda E} \tag{5.2.1}$$

Note that

$$\gamma(a, b\psi_\mu) = -\psi_\mu(a)\partial b - \psi_\mu(b)\partial a = 0 \tag{5.2.2}$$

(in $\mathcal{A}$), hence

$$h_{\mathfrak{g},\mathfrak{g}'}(a\psi'_\mu) = a h_{\mathfrak{g},\mathfrak{g}'}(\psi'_\mu) - \gamma(a,\psi'_\mu) = 0 \tag{5.2.3}$$

therefore

$$h_{\mathfrak{g},\mathfrak{g}'}(\psi''_\alpha) = h_{\mathfrak{g},\mathfrak{g}'}(A'^{-1\mu\alpha}\psi'_\mu) = 0 \tag{5.2.4}$$

It follows that

$$\mathfrak{a}(\psi''_\alpha) = 0 \tag{5.2.5}$$

5.3. Let us denote for brevity $h := h_{\mathfrak{g},\mathfrak{g}'},\ h' = h_{\mathfrak{g}',\mathfrak{g}''},\ h'' = h_{\mathfrak{g},\mathfrak{g}''}$.

We have

$$\begin{aligned} h(a\tau'_p) &= ah(\tau'_p) - \gamma(a,\tau'_p) = \\ &= ah^{pr}\omega_r + \tau_q(a)\partial g^{pq} + \tau_q(g^{pq})\partial a - g^{p\mu\mu}\partial a \end{aligned} \tag{5.3.1}$$

and

$$h(a\phi'_\gamma\psi'_\alpha) = -aA^{\gamma\beta}\partial A^{-1\beta\alpha} \tag{5.3.2}$$

Thus we get

$$\begin{aligned} h(\tau''_i) &= h(g'^{ip}\tau'_p + g'^{i\alpha\gamma}\phi'_\gamma\psi'_\alpha) = \\ &= g'^{ip}h^{pr}_\Omega\omega_r - g'^{ip}h^{pr}_E\omega_r + \tau_q(g'^{ip})\partial g^{pq} + \tau_q(g^{pq})\partial g'^{ip} - \\ &\quad - g^{p\mu\mu}\partial g'^{ip} - g'^{i\alpha\gamma}A^{\gamma\beta}\partial A^{-1\beta\alpha} \end{aligned} \tag{5.3.3}$$

It follows that

$$\mathfrak{a} = \mathfrak{a}_\Omega - \mathfrak{a}_E \tag{5.3.4}$$

where

$$\mathfrak{a}_\Omega(\tau''_i) = g'^{ip}h^{pr}_\Omega\omega_r + \tau_q(g'^{ip})\partial g^{pq} + \tau_q(g^{pq})\partial g'^{ip} + h'^{ip}_\Omega\omega'_p - h''^{ir}_\Omega\omega_r \tag{5.3.5}$$

and

$$\mathfrak{a}_E(\tau''_i) = g'^{ip}h^{pr}_E\omega_r + g^{p\mu\mu}\partial g'^{ip} + g'^{i\alpha\gamma}A^{\gamma\beta}\partial A^{-1\beta\alpha} + h'^{ip}_E\omega'_p - h''^{ir}_E\omega_r \tag{5.3.6}$$

The form $\mathfrak{a}_\Omega$ has already been computed in [GII], Section 6. Namely, by Theorem 6.4 (a) from *op. cit.*,

$$\mathfrak{a}_\Omega(\tau''_i) = \frac{1}{2} tr\{g'^{-1}\tau''_i(g')\tau''_s(g)g^{-1} - g'^{-1}\tau''_s(g')\tau''_i(g)g^{-1}\}\omega''_s \tag{5.3.7}$$

5.4. Let us compute $\mathfrak{a}_E(\tau''_i)$. Let us denote the five terms in the rhs of (5.3.6) by $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ and $\mathfrak{E}$. Thus,

$$\mathfrak{A} = g'^{ip}h_E^{pr}\omega_r = \frac{1}{2}g'^{ip}g^{pq}\tau_r(A^{-1\mu\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma\nu})A^{\nu\mu}\omega_r + g'^{ip}\tau_r(g^{p\nu\nu})\omega_r \tag{5.4.1}$$

$$\mathfrak{B} = g^{p\mu\mu}\partial g'^{ip} = g^{pq}\tau_q(A^{-1\mu\alpha})A^{\alpha\mu}\tau_r(g'^{ip})\omega_r \tag{5.4.2}$$

$$\mathfrak{C} = g'^{i\alpha\gamma}A^{\gamma\beta}\partial A^{-1\beta\alpha} = g'^{ip}\tau'_p(A'^{-1\alpha\mu})A'^{\mu\gamma}A^{\gamma\beta}\tau_r(A^{-1\beta\alpha})\omega_r \tag{5.4.3}$$

$$\mathfrak{D} = h'^{ij}_E\omega'_j = \frac{1}{2}g'^{is}\tau'_j(A'^{-1\mu\beta})A'^{\beta\gamma}\tau'_s(A'^{-1\gamma\nu})A'^{\nu\mu}g^{-1rj}\omega_r + \tau'_j(g'^{i\nu\nu})g^{-1rj}\omega_r =$$

$$= \frac{1}{2}g'^{is}\tau_r(A'^{-1\mu\beta})A'^{\beta\gamma}g^{sq}\tau_q(A'^{-1\gamma\nu})A'^{\nu\mu}\omega_r + \tau_r\{g'^{ip}g^{pq}\tau_q(A'^{-1\nu\mu})A^{\mu\nu}\}\omega_r \tag{5.4.4}$$

and

$$\mathfrak{E} = -h''^{ir}\omega_r = -\frac{1}{2}(g'g)^{iq}\tau_r\big((A'A)^{-1\mu\beta}\big)(A'A)^{\beta\gamma}\tau_q\big((A'A)^{-1\gamma\nu}\big)(A'A)^{\nu\mu}\omega_r - \tau_r(g''^{i\nu\nu}) =$$

$$= -\frac{1}{2}(g'g)^{iq}\{\tau_r(A^{-1\mu\sigma})A'^{-1\sigma\beta} + A^{-1\mu\sigma}\tau_r(A'^{-1\sigma\beta})\}A'^{\beta\rho}A^{\rho\gamma}\times$$

$$\times\{\tau_q(A^{-1\gamma\delta})A'^{-1\delta\nu} + A^{-1\gamma\delta}\tau_q(A'^{-1\delta\nu})\}A'^{\nu\epsilon}A^{\epsilon\mu}\omega_r - \tau_r(g''^{i\nu\nu}) =$$

$$= -\frac{1}{2}g'^{ip}g^{pq}\{\tau_r(A^{-1\mu\sigma})A^{\sigma\gamma}\tau_q(A^{-1\gamma\delta})A^{\delta\mu}\omega_r + \tau_r(A^{-1\mu\sigma})\tau_q(A'^{-1\sigma\nu})A'^{\nu\epsilon}A^{\epsilon\mu}\omega_r +$$

$$+\tau_r(A'^{-1\sigma\beta})A'^{\beta\rho}A^{\rho\gamma}\tau_q(A^{-1\gamma\sigma})\omega_r + \tau_r(A'^{-1\sigma\beta})A'^{\beta\rho}\tau_q(A'^{-1\rho\nu})A'^{\nu\sigma}\omega_r\} -$$

$$-\tau_r\{(g'g)^{iq}\tau_q\big((A'A)^{-1\nu\mu}\big)(A'A)^{\mu\nu}\}\omega_r \tag{5.4.5}$$

We see that $\mathfrak{A}1 = -\mathfrak{E}1$, $\mathfrak{C} = -2\mathfrak{E}2$, $\mathfrak{D}1 = -\mathfrak{E}4$. It is easy to see that $\mathfrak{A}2 + \mathfrak{D}2 + \mathfrak{E}5 = -\mathfrak{B}$.

Finally,

$$\mathfrak{C} + \mathfrak{E}2 = \frac{1}{2}tr\{A'^{-1}\tau''_i(A')\tau''_r(A)A^{-1}\}\omega''_r \tag{5.4.6}$$

and

$$\mathfrak{E}3 = -\frac{1}{2}tr\{A'^{-1}\tau''_r(A')\tau''_i(A)A^{-1}\}\omega''_r \tag{5.4.7}$$

So, we have proven

5.5. Lemma. *The form $\mathfrak{a}_E$ is given by*

$$\mathfrak{a}_E(\tau''_i) = \frac{1}{2}tr\{A'^{-1}\tau''_i(A')\tau''_r(A)A^{-1} - A'^{-1}\tau''_r(A')\tau''_i(A)A^{-1}\}\omega''_r \tag{5.5.1}$$

Combining 4.3 and 5.5 we get

5.6. Theorem. (a) *The cocycle* $\mathfrak{a}_{\mathfrak{g},\mathfrak{g}',\mathfrak{g}''}$ *is given by*

$$\mathfrak{a}_{\mathfrak{g},\mathfrak{g}',\mathfrak{g}''}(\tau_i'') = \frac{1}{2}tr\{g'^{-1}\tau_i''(g')\tau_r''(g)g^{-1} - g'^{-1}\tau_r''(g')\tau_i''(g)g^{-1}\}\omega_r'' -$$

$$-\frac{1}{2}tr\{A'^{-1}\tau_i''(A')\tau_r''(A)A^{-1} - A'^{-1}\tau_r''(A')\tau_i''(A)A^{-1}\}\omega_r'' \qquad (5.6.1)$$

(b) *The* 3*-form* $\mathfrak{b}_{\mathfrak{g},\mathfrak{g}'}$ *is given by*

$$\mathfrak{b}_{\mathfrak{g},\mathfrak{g}'}(\tau_i',\tau_j') = -\frac{1}{2}tr\{g^{-1}\tau_i'(g)g^{-1}\tau_j'(g)g^{-1}\tau_r'(g) - g^{-1}\tau_j'(g)g^{-1}\tau_i'(g)g^{-1}\tau_r'(g)\}\omega_r' +$$

$$+\frac{1}{2}tr\{A^{-1}\tau_i'(A)A^{-1}\tau_j'(A)A^{-1}\tau_r'(A) - A^{-1}\tau_j'(A)A^{-1}\tau_i'(A)A^{-1}\tau_r'(A)\}\omega_r' \qquad (5.6.2)$$

5.7. Lemma. *Let* $E^* = Hom_A(E,A)$ *be the dual module. We have*

$$(\mathfrak{a}_{E^*},\mathfrak{b}_{E^*}) = (\mathfrak{a}_E,\mathfrak{b}_E) \qquad (5.7.1)$$

Indeed, this follows from the easy identities

$$tr\{A^t\tau_i((A^t)^{-1})A^t\tau_j((A^t)^{-1})A^t\tau_r((A^t)^{-1})\} = -tr\{A^{-1}\tau_r(A)A^{-1}\tau_j(A)A^{-1}\tau_i(A)\} \qquad (5.7.2)$$

and

$$tr\{A^t\tau_i((A^t)^{-1}\tau_j((B^t)^{-1})B^t\} = tr\{A^{-1}\tau_i(A)\tau_j(B)B^{-1}\} \qquad (5.7.3)$$

5.8. Let us pass to the global situation. Let X be a smooth variety over k and E be a vector bundle over X. As in [GII], we define the gerbe $\mathfrak{D}_{\Lambda E}$ of chiral differential operators on ΛE over X.

Its characteristic class $c(\mathfrak{D}_{\Lambda E})$ will belong to the second hypercohomology $H^2(X;\Omega^{[2,3\rangle}_{\Lambda E})$ (in obvious notations). Recall that we have a canonical imbedding of de Rham complexes

$$\Omega^{\cdot}_X \hookrightarrow \Omega^{\cdot}_{\Lambda E} \qquad (5.8.1)$$

In [GII], 7.6 we have defined the "Atiyah-Chern-Simons" characteristic class $c(E) \in H^2(X;\Omega^{[2,3\rangle}_X)$; let us denote by $c(E)_{\Lambda E}$ its image in $H^2(X;\Omega^{[2,3\rangle}_{\Lambda E})$.

The theorem below is an immediate consequence Theorem 5.6 and Lemma 5.7.

5.9. Theorem. *The class* $c(\mathfrak{D}_{\Lambda E})$ *is equal to*

$$c(\mathfrak{D}_{\Lambda E}) = c(\Theta_X) - c(E) = c(\Omega^1_X) - c(E) \qquad (5.9.1)$$

where Θ_X *is the tangent bundle.*

§6. Chiral de Rham complex

6.1. Let us return to the local situation 3.1, 4.1. Let E be equal to the module of vector fields T. Given a base $\{\bar{\tau}_i\}$ consisting of commuting vector fields, we get a frame $\mathfrak{g} = \{\bar{\tau}_i; \phi_i := \bar{\tau}_i\}$ of (A, E). Let us call such frames *natural*.

Let $\mathfrak{g}, \mathfrak{g}'$ be two natural frames, with transition matrices as in 3.2. By definition, $(A^{rs}) = (g^{rs})$. Therefore the coefficients $g^{i\alpha\gamma}$ (3.2.2) are given by

$$g^{i\alpha\gamma} = g^{iq}\tau_q(g^{-1\alpha\mu})g^{\mu\gamma} = g^{iq}\tau_\alpha(g^{-1q\mu})g^{\mu\gamma} =$$

$$= -\tau_\alpha(g^{iq})g^{-1q\mu}g^{\mu\gamma} = -\tau_\alpha(g^{i\gamma}) \tag{6.1.1}$$

where we have used (3.3.3). Consequently the function h_E (4.1.6) is given by

$$h_E^{ij} = -\tau_j\tau_\nu(g^{i\nu}) + \frac{1}{2}g^{iq}\tau_j(g^{-1\mu\beta})g^{\beta\gamma}\tau_q(g^{-1\gamma\nu})g^{\nu\mu} \tag{6.1.2}$$

The second summand is equal to

$$-\frac{1}{2}g^{iq}\tau_j(g^{-1\mu\beta})\tau_q(g^{\beta\gamma})g^{-1\gamma\nu}g^{\nu\mu} = -\frac{1}{2}g^{iq}\tau_j(g^{-1\mu\beta})\tau_q(g^{\beta\mu}) =$$

$$= -\frac{1}{2}g^{iq}\tau_\mu(g^{-1j\beta})\tau_q(g^{\beta\mu}) = -\frac{1}{2}\tau_\mu(g^{-1j\beta})g^{\beta q}\tau_q(g^{i\mu}) = \frac{1}{2}g^{-1j\beta}\tau_\mu(g^{\beta q})\tau_q(g^{i\mu}) \tag{6.1.3}$$

We see that the first summand in (6.1.2) is equal to minus the first summand of (4.1.5), and second summand of (6.1.2) is equal to the second summand of (4.1.5). Thus

$$h^{ij} = 2\tau_p\tau_j(g^{ip}) \tag{6.1.4}$$

If $\mathfrak{g}, \mathfrak{g}', \mathfrak{g}''$ are natural frames of (A, T) then Theorem 5.6 says that $\mathfrak{a}_{\mathfrak{g},\mathfrak{g}',\mathfrak{g}''} = \mathfrak{b}_{\mathfrak{g},\mathfrak{g}'} = 0$. This means that

the chiral superalgebroid $\mathcal{A}_{\Lambda T;\mathfrak{g}}$ does not depend, up to a canonical isomorphism, on the choice of the base $\{\bar{\tau}_i\}$. In other words, we have a canonically defined chiral algebroid $\mathcal{A}_{\Lambda T}$.

Passing to chiral envelopes, we get a canonically defined chiral (vertex) superalgebra $D^{ch}_{\Lambda T}$ of *chiral differential operators on T.*

It follows that

for each smooth variety X we have a canonically defined sheaf of chiral superalgebras $\mathcal{D}^{ch}_{\Lambda\Theta_X}$. These Zariski sheaves form in fact a sheaf in the étale topology.

The gluing functions for this sheaf are given explicitly by (6.1.4).

6.2. Lemma. *In the situation* 4.1, *consider the function* h_E

$$h_E^{ij} = \tau_j(g^{i\nu\nu}) + \frac{1}{2}g^{iq}\tau_j(A^{-1\mu\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma\nu})A^{\nu\mu} \tag{6.2.1}$$

The function h_{E^*} associated with the dual module E^* is given by

$$h^{ij}_{E^*} = -\tau_j(g^{i\nu\nu}) + \frac{1}{2} g^{iq}\tau_j(A^{-1\mu\beta})A^{\beta\gamma}\tau_q(A^{-1\gamma\nu})A^{\nu\mu} \tag{6.2.2}$$

This follows from the identities

$$tr\{\tau_i(A^t)A^{-1}\} = -tr\{\tau_i(A^{-1})A\} \tag{6.2.3}$$

and

$$tr\{\tau_i(A^t)A^{-1}\tau_j(A^t)A^{-1}\} = tr\{\tau_i(A^{-1})A\tau_j(A^{-1})A\} \tag{6.2.4}$$

(cf. (5.7.2), (5.7.3)).

6.3. Let E be the module of 1-forms $\Omega = \Omega^1_{A/k}$; its exterior algebra is the de Rham algebra of differential forms $\Omega^{\cdot} = \Omega^{\cdot}_{A/k}$. Frames of the form $\mathfrak{g} = \{\bar{\tau}_i, \phi_i := \omega_i\}$ will be called natural.

If $\mathfrak{g}, \mathfrak{g}'$ are natural frames then formulas (6.1.2) and (6.1.3), together with the previous lemma, show that $h^{ij}_E = h^{ij}_\Omega$ where h_Ω is given by (4.1.5). (Of course one easily checks this directly.) This explains the notation for h^{ij}_Ω).

In other words, we arrive at an interesting conclusion.

6.4. Theorem. *The matrices $h = (h^{ij})$ defined in* 4.1 *are equal to* 0 *if $E = \Omega$ and frames $\mathfrak{g}, \mathfrak{g}'$ are natural.*

6.4.1. *Warning.* The functions $h_{\mathfrak{g},\mathfrak{g}'}$ are *non*zero since they are not linear.

6.5. On the other hand, Theorem 5.6 together Lemma 5.7 say that $\mathfrak{a}_E$ and $\mathfrak{b}_E$ are 0 for $E = \Omega$ (for natural frames $\mathfrak{g}, \mathfrak{g}', \mathfrak{g}''$ of (A, Ω)). This gives us a *canonically defined* chiral superalgebroid $\mathcal{A}_{\Omega^{\cdot}}$. Its vertex envelope will be denoted $D^{ch}_{\Omega^{\cdot}}$ and called *the chiral algebra of differential operators on* $\Omega^{\cdot}$.

This implies

6.6. Theorem. *For each smooth variety X the construction* 6.2 - 6.5 *gives a canonically defined sheaf of chiral superalgebras $\mathcal{D}^{ch}_{\Omega^{\cdot}_X}$. These Zariski sheaves form a sheaf in the étale topology.*

6.7. The de Rham differential may be considered as an odd first order differential operator acting on $\Omega^{\cdot}_X$ (and commuting with itself).

In coordinates, if $\mathfrak{g} = \{\bar{\tau}_i; \phi_i := \omega_i\}$ is a natural frame of (A, Ω), it is given by

$$Q^{cl}_{\mathfrak{g}} = \phi_i\tau_i \tag{6.7.1}$$

(cl is for "classical"). Let us check the independence of (6.7.1) on the choice of a frame. Let $\mathfrak{g}' = \{\bar{\tau}'_i; \omega'_i\}$ be another natural frame, $\bar{\tau}'_i = g^{ij}\bar{\tau}_i$; $\omega'_i = g^{-1pi}\omega_p$ (cf. (3.2.4)).

Using (3.2.1) we have

$$Q^{cl}_{\mathfrak{g}'} = \phi'_i\tau'_i = g^{-1pi}\phi_p\{g^{iq}\tau_q + g^{irs}\phi_s\psi_r\} \tag{6.7.2}$$

where

$$g^{ipq} = g^{ir}\tau_r(g^{sp})g^{-1qs} = g^{sr}\tau_r(g^{ip})g^{-1qs} = \tau_q(g^{ip}) \tag{6.7.3}$$

(we have used (3.3.1)). So,

$$Q^{cl}_{\mathfrak{g}'} = \phi_p\tau_p + g^{-1pi}g^{irs}\phi_s\psi_r = Q^{cl}_{\mathfrak{g}} + g^{-1pi}\tau_s(g^{ir})\phi_p\phi_s\psi_r \tag{6.7.4}$$

Note that

$$g^{-1pi}\tau_s(g^{ir}) = -\tau_s(g^{-1pi})g^{ir}$$

which is symmetric under the permutation of p with s, due to (3.3.3); therefore the second summand in (6.7.4) is zero, i.e. $Q^{cl}_{\mathfrak{g}} = Qcl_{\mathfrak{g}'}$.

Thus, Q^{cl} is a correctly defined odd element of $D_{\Omega^\cdot}$. It is obvious from (6.7.1) that $[Q^{cl}, Q^{cl}] = 0$.

6.8. Let us investigate the chiral counterpart of Q^{cl}. Let us define an odd element $Q_{\mathfrak{g}}$ (of conformal weight 1) of the vertex superalgebra $D^{ch}_{\Omega^\cdot;\mathfrak{g}} := U\mathcal{A}_{\Omega^\cdot;\mathfrak{g}}$ by

$$Q_{\mathfrak{g}} = \phi_{i(-1)}\tau_i \tag{6.8.1}$$

Let $\mathfrak{g}'$ be another natural frame as in 6.7. Due to Theorem 6.4, the element $Q_{\mathfrak{g}'}$ goes under the canonical isomorphism $D^{ch}_{\Omega^\cdot;\mathfrak{g}'} = D^{ch}_{\Omega^\cdot;\mathfrak{g}}$ to

$$Q_{\mathfrak{g}'} = \phi'_{i(-1)}\tau'_i = \phi'_i\tau'_i - \gamma(\phi'_i, \tau'_i) \tag{6.8.2}$$

cf. [GII], (3.3.1).

6.9. Lemma. *We have (in $D_{\Omega^\cdot;\mathfrak{g}}$)*

$$\gamma(\phi'_i, \tau'_i) = -\partial\big\{tr\big(\tau_r(g)g^{-1}\big)\phi_r\big\} \tag{6.9.1}$$

6.10. Before the proof, let us write down useful formulas

$$\gamma(a\phi_r, b\tau_i) = -\tau_i(a)\phi_r\partial b - \tau_i(b)\partial(a\phi_r) \tag{6.10.1}$$

and

$$\gamma(a\phi_r, b\phi_s\psi_p) = -\delta_{rp}a\partial(b\phi_s) + \delta_{sp}b\partial(a\phi_r) \tag{6.10.2}$$

6.11. *Proof of* 6.9. We have

$$\gamma(\phi'_i, \tau'_i) = \gamma(g^{-1qi}\phi_q, g^{ip}\tau_p + g^{isr}\phi_r\psi_s)$$

where

$$\gamma(g^{-1qi}\phi_q, g^{ip}\tau_p) = -\tau_p(g^{-1qi})\phi_q\partial g^{ip} - \tau_p(g^{ip})\partial(g^{-1qi}\phi_q) \tag{6.11.1}$$

and

$$\gamma(g^{-1qi}\phi_q, g^{isr}\phi_r\psi_s) = -g^{-1qi}\partial(g^{iqr}\phi_r) + g^{irr}\partial(g^{-1qi})\phi_q \tag{6.11.2}$$

Since $g^{irr} = \tau_r(g^{ir})$, the second summands in (6.11.1) and (6.11.2) cancel out. On the other hand, the first term in (6.11.1) is equal to

$$-\tau_q(g^{-1ri})\tau_s(g^{iq})\omega_s\phi_r = -\tau_r(g^{-1si})\tau_s(g^{iq})\omega_s\phi_r =$$

$$= g^{-1qa}\tau_r(g^{ab})g^{-1bi}\tau_s(g^{iq})\omega_s\phi_r = -\tau_r(g^{ab})\tau_s(g^{-1ba})\omega_s\phi_r = -\tau_r(g^{ab})\partial(g^{-1ba})\phi_r$$

Therefore

$$\gamma(\phi'_i, \tau'_i) = -\tau_r(g^{ab})\partial(g^{-1ba})\phi_r - g^{-1ba}\partial\big\{\tau_r(g^{ab})\phi_r\big\} = -\partial\big\{\tau_r(g^{ab})g^{-1ba}\phi_r\big\},$$

QED.

6.12. From (6.8.1) we have $Q_{\mathfrak{g}} = \phi_i\tau_i$, and from 6.7 $\phi'_i\tau'_i = \phi_i\tau_i$. Therefore, (6.8.2) and Lemma 6.9 imply

6.13. Theorem. *We have*

$$Q_{\mathfrak{g}'} = Q_{\mathfrak{g}} + \partial\big\{tr\big(\tau_r(g)g^{-1}\big)\phi_r\big\} \tag{6.13.1}$$

6.14. Consider the field $Q_{\mathfrak{g}}(z)$ acting on the vertex algebra $D^{ch}_{\Omega^\cdot}$. Due to (6.13.1), its zeroth component $Q_{\mathfrak{g}0}$ does not depend on the choice of the frame $\mathfrak{g}$. Therefore we get a canonical operator Q_0 acting on $D^{ch}_{\Omega^\cdot}$.

Since it is a zeroth component of a field, it is a derivation of the vertex algebra, and it is obvious from the local definition (6.8.1) that $[Q_0, Q_0] = 0$.

Consequently, for each smooth variety X we get a canonical odd derivation Q_{0X} of the sheaf $\mathcal{D}^{ch}_{\Omega^\cdot_X}$, such that $[Q_{0X}, Q_{0X}] = 0$. The pair $(\mathcal{D}^{ch}_{\Omega^\cdot_X}, Q_{0X})$ is the *chiral de Rham complex* from [MSV].

Our Theorem 6.13 is a version of *op. cit.*, (4.1c).

6.15. In the situation 6.4, consider an even element $J_{\mathfrak{g}}$ of conformal weight 1 of the algebra $D^{ch}_{\Omega^\cdot;\mathfrak{g}}$, given by

$$J_{\mathfrak{g}} = \phi_{i(-1)}\psi_i = \phi_i\psi_i \tag{6.15.1}$$

After a change of frame as in *loc. cit.*, we get an element

$$J_{\mathfrak{g}'} = \phi'_{i(-1)}\psi'_i = \phi'_i\psi'_i - \gamma(\phi'_i, \psi'_i) \tag{6.15.2}$$

where we have again used Theorem 6.4. We have

$$\phi'_i\psi'_i = g^{-1pi}\phi_p g^{iq}\psi_q = \phi_p\psi_p = J_{\mathfrak{g}}$$

(see (3.2.3)). On the other hand, by (3.4.3)

$$\gamma(\phi'_i, \psi'_i) = \gamma(g^{-1pi}\phi_p, g^{iq}\psi_q) = \delta_{pq}g^{-1pi}\partial g^{iq} = g^{-1pi}\partial g^{ip} = tr(g^{-1}\partial g)$$

Thus

$$J_{\mathfrak{g}'} = J_{\mathfrak{g}} - tr(g^{-1}\partial g) \tag{6.15.3}$$

6.16. Consider an odd element $G_{\mathfrak{g}}$ of conformal weight 2 given by

$$G_{\mathfrak{g}} = \psi_{i(-1)}\omega_i \tag{6.16.1}$$

In the frame $\mathfrak{g}'$,

$$G_{\mathfrak{g}'} = (g^{iq}\psi_q)_{(-1)}(g^{-1si}\omega_s)$$

Note that $\psi_{q(j)}a = 0$ for $j \geq 0$ (everything happens in D), hence it follows from the commutativity formula (1.3.1) that

$$a\psi_q = a_{(-1)}\psi_q = \psi_{q(-1)}a \tag{6.16.2}$$

Therefore by "associativity" (1.2.5)

$$(a\psi_q)_{(-1)}(b\omega_s) = (\psi_{q(-1)}a)_{(-1)}(b\omega_s) = \psi_{q(-1)}a_{(-1)}b\omega_s = \psi_{q(-1)}(ab\omega_s) \tag{6.16.3}$$

Therefore

$$G_{\mathfrak{g}'} = \psi_{q(-1)}g^{iq}g^{-1si}\omega_s = \psi_{s(-1)}\omega_s = G_{\mathfrak{g}} \tag{6.16.6}$$

6.17. Let us investigate the *Virasoro element*. Define an even element $L_{\mathfrak{g}}$ of conformal weight 2 by

$$L_{\mathfrak{g}} = L_{(b)\mathfrak{g}} + L_{(f)\mathfrak{g}} \tag{6.17.1}$$

where

$$L_{(b)\mathfrak{g}} = \omega_{i(-1)}\tau_i \tag{6.17.2}$$

((b) is for "bosonic") and

$$L_{(f)\mathfrak{g}} = \rho_{i(-1)}\psi_i \tag{6.17.3}$$

((f) is for "fermionic"), cf. [MSV], (2.3a).

6.18. We have

$$(a\omega_s)_{(-1)}(b\tau_p) = \omega_{s(-1)}\big\{ab\tau_p + \tau_p(a)\partial b + \tau_p(b)\partial a\big\} - \partial\omega_{s(-1)}b\tau_p(a) \tag{6.18.1}$$

Indeed, it follows from "associativity" (1.2.5) that

$$(a\omega_s)_{(-1)}(b\tau_p) = (\omega_{s(-1)}a)_{(-1)}(b\tau_p) = \omega_{s(-1)}a_{(-1)}(b\tau_p) + \omega_{s(-2)}a_{(0)}(b\tau_p) +$$

$$+a_{(-2)}\omega_{s(0)}(b\tau_p)$$

Next,

$$a_{(-1)}(b\tau_p) = ab\tau_p - \gamma(a, b\tau_p) = ab\tau_p + \tau_p(a)\partial b + \tau_p(b)\partial a$$

(see [GII] (3.3.1));

$$a_{(0)}(b\tau_p) = -b\tau_{p(0)}a = -b\tau_p(a)$$

(see [GII] (3.3.2)). Finally

$$\omega_{s(0)}(b\tau_p) = -(b\tau_p)_{(0)}\omega_s + \partial\langle b\tau_p, \omega_s\rangle$$

where

$$(b\tau_p)_{(0)}\omega_s = (b\tau_p)(\omega_s) = \langle \tau_p, \omega_s\rangle \partial b = \delta_{ps}\partial b$$

by (1.1.3), since

$$\tau_p(\omega_s) = 0, \tag{6.18.2}$$

and $\langle b\tau_p, \omega_s\rangle = b\delta_{ps}$. This implies

$$\omega_{s(0)}(b\tau_p) = 0 \tag{6.18.3}$$

Formula (6.18.1) follows.

6.19. We have

$$(a\omega_s)_{(-1)}(b\phi_\alpha\psi_\beta) = \omega_{s(-1)}\{ab\phi_\alpha\psi_\beta - \delta_{\alpha\beta}b\partial a\} \tag{6.19.1}$$

Indeed, by commutativity and "associativity" (1.2.5)

$$(a\omega_s)_{(-1)}(b\phi_\alpha\psi_\beta) = (\omega_{s(-1)}a)_{(-1)}(b\phi_\alpha\psi_\beta) = \omega_{s(-1)}a_{(-1)}(b\phi_\alpha\psi_\beta)$$

On the other hand

$$a_{(-1)}(b\phi_\alpha\psi_\beta) = ab\phi_\alpha\psi_\beta - \gamma(a, b\phi_\alpha\psi_\beta) = ab\phi_\alpha\psi_\beta - \delta_{\alpha\beta}b\partial a,$$

see (3.4.2). This implies (6.19.1).

6.20. We have

$$L_{(b)\mathfrak{g}'} = L_{(b)\mathfrak{g}} + \omega_{s(-1)}\{\tau_p(g^{-1si})\partial g^{ip} + g^{-1si}\tau_\beta(g^{i\alpha})\phi_\beta\psi_\alpha\} -$$

$$-\omega_{s(-2)}g^{ip}\tau_p(g^{-1si}) \tag{6.20.1}$$

Indeed, due to Theorem 6.4

$$L_{(b)\mathfrak{g}'} = (g^{-1si}\omega_s)_{(-1)}\{g^{ip}\tau_p + g^{i\alpha\beta}\phi_\beta\psi_\alpha\}$$

According to (6.18.1)

$$(g^{-1si}\omega_s)_{(-1)}(g^{ip}\tau_p) = \omega_{s(-1)}\{g^{-1si}g^{ip}\tau_p + \tau_p(g^{-1si})\partial g^{ip} + \tau_p(g^{ip})\partial g^{-1si}\} -$$

$$-\omega_{s(-2)}g^{ip}\tau_p(g^{-1si})$$

By (6.18.2)

$$(g^{-1si}\omega_s)_{(-1)}(g^{i\alpha\beta}\phi_\beta\psi_\alpha) = (g^{-1si}\omega_s)_{(-1)}(\tau_\beta(g^{i\alpha})\phi_\beta\psi_\alpha) =$$

$$= \omega_{s(-1)}\{g^{-1si}\tau_\beta(g^{i\alpha})\phi_\beta\psi_\alpha - \delta_{\alpha\beta}\tau_\beta(g^{i\alpha})\partial g^{-1si}\}$$

The third term in the first expression cancels out the second term in the second one, and we get (6.20.1).

6.21. We have

$$(a\rho_\mu)_{(-1)}(b\psi_\nu) = \rho_{\mu(-1)}ab\psi_\nu \tag{6.21.1}$$

Indeed,

$$(a\rho_\mu)_{(-1)}(b\psi_\nu) = (\rho_{\mu(-1)}a)_{(-1)}(b\psi_\nu) = \rho_{\mu(-1)}ab\psi_\nu + a_{(-2)}\rho_{\mu(0)}b\psi_\nu$$

and by commutativity

$$\rho_{\mu(0)}b\psi_\nu = (b\psi_\nu)(\rho_\mu) - \partial\big\{(b\psi_\nu)_{(1)}\rho_\mu\big\} = \delta_{\nu\mu}\partial b - \delta_{\nu\mu}\partial b = 0,$$

cf. (6.18.3). This implies (6.21.1).

6.22. We have

$$(a\phi_\gamma\omega_i)_{(-1)}(b\psi_\nu) = \omega_{i(-1)}\big\{ab\phi_\gamma\psi_\nu - \delta_{\gamma\nu}a\partial b\big\} + \delta_{\nu\gamma}\omega_{i(-2)}ab \tag{6.22.1}$$

Indeed,

$$(a\phi_\gamma\omega_i)_{(-1)}(b\psi_\nu) = (\omega_{i(-1)}a\phi_\gamma)_{(-1)}(b\psi_\nu) = \omega_{i(-1)}(a\phi_\gamma)_{(-1)}b\psi_\nu + \omega_{i(-2)}(a\phi_\gamma)_{(0)}b\psi_\nu$$

where

$$(a\phi_\gamma)_{(-1)}b\psi_\nu = a\phi_\gamma b\psi_\nu - \gamma(a\phi_\gamma, b\psi_\nu) = ab\phi_\gamma\psi_\nu - \delta_{\gamma\nu}a\partial b,$$

cf. (3.4.3), and

$$(a\phi_\gamma)_{(0)}b\psi_\nu = (b\psi_\nu)(a\phi_\gamma) = ab\delta_{\nu\gamma}$$

6.23. We have

$$L_{(f)\mathfrak{g}'} = L_{(f)\mathfrak{g}} + \omega_{i(-1)}\big\{\tau_i(g^{-1\gamma\alpha})g^{\alpha\nu}\phi_\gamma\psi_\nu - \tau_i(g^{-1\gamma\alpha})\partial g^{\alpha\gamma}\big\} + \omega_{i(-2)}\tau_i(g^{-1\gamma\alpha})g^{\alpha\gamma} \tag{6.23.1}$$

Indeed,

$$L_{(f)\mathfrak{g}'} = \rho'_{\alpha(-1)}\psi'_\alpha = \big\{g^{-1\mu\alpha}\rho_\mu + \tau_i(g^{-1\gamma\alpha})\phi_\gamma\omega_i\big\}_{(-1)}(g^{\alpha\nu}\psi_\nu)$$

By (6.21.1)

$$(g^{-1\mu\alpha}\rho_\mu)_{(-1)}(g^{\alpha\nu}\psi_\nu) = \rho_{\mu(-1)}g^{-1\mu\alpha}g^{\alpha\nu}\psi_\nu = L_{(f)\mathfrak{g}}$$

and by (6.22.1)

$$(\tau_i(g^{-1\gamma\alpha})\phi_\gamma\omega_i)_{(-1)}(g^{\alpha\nu}\psi_\nu) = \omega_{i(-1)}\big\{\tau_i(g^{-1\gamma\alpha})g^{\alpha\nu}\phi_\gamma\psi_\nu - \tau_i(g^{-1\gamma\alpha})\partial g^{\alpha\gamma}\big\} +$$

$$+\omega_{i(-2)}\tau_i(g^{-1\gamma\alpha})g^{\alpha\gamma}$$

6.24. Comparing (6.20.1) and (6.23.1) we see easily that

$$L_{\mathfrak{g}'} = L_{(b)\mathfrak{g}'} + L_{(f)\mathfrak{g}'} = L_{(b)\mathfrak{g}} + L_{(f)\mathfrak{g}} = L_{\mathfrak{g}} \tag{6.24.1}$$

Let us collect our computations of transformation rules.

6.25. Theorem. *Let $\mathfrak{g}, \mathfrak{g}'$ be two natural frames of (A, Ω). Consider 4 elements of the vertex superalgebra $D^{ch}_{\Omega^\cdot;\mathfrak{g}}$ given by*

$$Q_{\mathfrak{g}} = \phi_{i(-1)}\tau_i \tag{6.25.1}$$

(an odd element of conformal weight 1)

$$J_{\mathfrak{g}} = \phi_{i(-1)}\psi_i \tag{6.25.2}$$

(an even element of conformal weight 1)

$$G_{\mathfrak{g}} = \psi_{i(-1)}\omega_i \tag{6.25.3}$$

(an odd element of conformal weight 2) and

$$L_{\mathfrak{g}} = \omega_{i(-1)}\tau_i + \rho_{i(-1)}\psi_i \tag{6.25.4}$$

(an even element of conformal weight 2).

After the canonical identification $D^{ch}_{\Omega^{\cdot};\mathfrak{g}'} = D^{ch}_{\Omega^{\cdot};\mathfrak{g}}$ these elements are transformed as follows

$$Q_{\mathfrak{g}'} = Q_{\mathfrak{g}} + \partial\{tr(g^{-1}\tau_r(g))\phi_r\} \tag{6.25.5}$$

$$J_{\mathfrak{g}'} = J_{\mathfrak{g}} - tr(g^{-1}\partial g) \tag{6.25.6}$$

$$G_{\mathfrak{g}'} = G_{\mathfrak{g}} \tag{6.25.7}$$

and

$$L_{\mathfrak{g}'} = L_{\mathfrak{g}} \tag{6.25.8}$$

This is a version of [MSV], Theorem 4.2.

§7. Poincaré-Birkhoff-Witt

7.1 Let X be a smooth variety and $\mathcal{D}^{ch}_{\Omega^{\cdot}_X}$ be the sheaf discussed in the previous section, cf. Theorem 6.6. It is a sheaf of $\mathbb{Z}_{\geq 0}$-graded vertex algebras, so

$$\mathcal{D}^{ch}_{\Omega^{\cdot}_X} = \oplus_{n\in\mathbb{Z}_{\geq 0}}\ \mathcal{D}^{ch}_{\Omega^{\cdot}_X;n} \tag{7.1.1}$$

where $\mathcal{D}^{ch}_{\Omega^{\cdot}_X;n}$ denotes the component of conformal weight n.

According to Theorem 6.25 (see (6.25.8)) we have a canonical global section L of $\mathcal{D}^{ch}_{\Omega^{\cdot}_X;2}$. Let $L(z) = \sum\ L_n z^{-n-2}$ be the corresponding field.

7.1.1. Claim. *A local section $\alpha \in \mathcal{D}^{ch}_{\Omega^{\cdot}_X}$ belongs to $\mathcal{D}^{ch}_{\Omega^{\cdot}_X;n}$ if and only if $L_0(\alpha) = n\alpha$.*

In a uniform notation $L(z) = \sum\ L_{(n)}z^{-n-1}$ we have $L_0 = L_{(1)}$. We shall check 7.1.1 simultaneously with

7.1.2. Claim. *The operator $L_{-1} = L_{(0)}$ coincides with the canonical derivation ∂ of the vertex algebra $\mathcal{D}^{ch}_{\Omega^{\cdot}_X}$.*

Both statements are local, so we may assume we are in the local situation 6.17. Note that the operator $L_{(0)}$ is a derivation with respect to the operation ${}_{(-1)}$:

$$L_{(0)}(x_{(-1)}y) = (L_{(0)}x)_{(-1)}y + x_{(-1)}L_{(0)}y \tag{7.1.2}$$

Therefore it suffices to check 7.1.2 on the generators $a, \tau_i, \omega_i, \psi_i, \rho_i$ of the vertex algebroid $\mathcal{A}_{\Omega^\cdot;\mathfrak{g}}$, which is done by a simple explicit computation.

It follows from the associativity formula (1.2.4) that

$$L_{(1)}y_{(-1)}z = (L_{(1)}y)_{(-1)}z + y_{(-1)}L_{(1)}z + L_{(0)}y_{(0)}z - y_{(0)}L_{(0)}z =$$

$$= (L_{(1)}y)_{(-1)}z + y_{(-1)}L_{(1)}z \tag{7.1.3}$$

since

$$L_{(0)}y_{(0)}z - y_{(0)}L_{(0)}z = \partial(y_{(0)}z) - y_{(0)}\partial z = 0$$

In other words, $L_{(1)}$ is a derivation of ${}_{(-1)}$. Therefore it suffices to check 7.1.1 on the generators of $\mathcal{A}_{\Omega^\cdot;\mathfrak{g}}$ as above, which is straightforward.

7.2. In the local situation 6.1, consider the local algebra $D^{ch}_{\Omega^\cdot;\mathfrak{g}} = U\mathcal{A}_{\Omega^\cdot;\mathfrak{g}}$. Let us introduce a $\mathbb{Z}$-grading

$$D^{ch}_{\Omega^\cdot;\mathfrak{g}} = \oplus_{p\in Z}\ D^{ch;p}_{\Omega^\cdot;\mathfrak{g}} \tag{7.2.1}$$

to be called *fermionic charge*. For an element $x \in D^{ch}_{\Omega^\cdot;\mathfrak{g}}$ let us denote its fermionic charge (to be defined) by $F(x) \in \mathbb{Z}$. It is defined uniquely by the following conditions:

(a) $F(a) = F(\tau_i) = F(\omega_i) = 0;\ F(\phi_i) = F(\rho_i) = -F(\psi_i) = 1;$

(b) $F(x_{(-1)}y) = F(x) + F(y)$.

Due to the transformation formulas (3.2.1) - (3.2.5) this grading is obviously preserved under a change of frames. Therefore in the situation of 7.1 the sheaf $\mathcal{D}^{ch}_{\Omega^\cdot_X}$ gets a canonical $\mathbb{Z}$-grading

$$\mathcal{D}^{ch}_{\Omega^\cdot_X} = \oplus_{p\in\mathbb{Z}}\ \mathcal{D}^{ch;p}_{\Omega^\cdot_X} \tag{7.2.2}$$

Note that parity is equal to fermionic charge modulo 2.

Here is another way to define the grading (7.2.2). First notice a simple

7.2.1. Lemma. *Let $\mathcal{A} = (A, T, \Omega, \partial, \dots)$ be a vertex (super)algebroid. For every invertible element $a \in A$ the operator $(a^{-1}\partial a)_{(0)}$ acting on $U\mathcal{A}$ is trivial.*

Proof. Obviously this operator is trivial on $A = U\mathcal{A}_0$. Let $x \in U\mathcal{A}_1$ and $\tau \in T$ be its image under the canonical projection $U\mathcal{A}_1 \longrightarrow T$. We have

$$(a^{-1}\partial a)_{(0)}x = -x_{(0)}a^{-1}\partial a + \partial(x_{(1)}a^{-1}\partial a) = -\tau(a^{-1}\partial a) + \partial\langle\tau, a^{-1}\partial a\rangle =$$

$$= a^{-2}\tau(a)\partial a - a^{-1}\partial\tau(a) + \partial(a^{-1}\tau(a)) = 0,$$

so $(a^{-1}\partial a)_{(0)}$ is trivial on $U\mathcal{A}_1$. Therefore it is trivial on the whole algebra $U\mathcal{A}$ since $(?)_{(0)}$ is a derivation of the operation ${}_{(-1)}$. $\triangle$

Applying this lemma to $a = det(g)$ in the formula (6.25.6) we see that the component $J_{0;\mathfrak{g}}$ of the field $J_{\mathfrak{g}}(z) = \sum J_{n;\mathfrak{g}} z^{-n-1}$ is preserved under the change of frames. Consequently it gives rise to a well defined endomorphism J_0 of the sheaf $\mathcal{D}^{ch}_{\Omega^{\cdot}_X}$.

7.2.2. Claim. *A local section $\alpha \in \mathcal{D}^{ch}_{\Omega^{\cdot}_X}$ belongs to $\mathcal{D}^{ch;p}_{\Omega^{\cdot}_X}$ if and only if $J_0(\alpha) = p\alpha$.*

Indeed, the function $F(\alpha)$ defined by $J_0(\alpha) = F(\alpha)\alpha$ obviously satisfies the condtions (a) and (b) above.

7.3. The two gradings (7.1.1) and (7.2.2) are compatible: if we denote

$$\mathcal{D}^{ch;p}_{\Omega^{\cdot}_X;n} := \mathcal{D}^{ch}_{\Omega^{\cdot}_X;n} \cap \mathcal{D}^{ch;p}_{\Omega^{\cdot}_X} \tag{7.3.1}$$

then

$$\mathcal{D}^{ch}_{\Omega^{\cdot}_X} = \oplus_{n\in\mathbb{Z}_{\geq 0};\ p\in\mathbb{Z}}\ \mathcal{D}^{ch;p}_{\Omega^{\cdot}_X;n} \tag{7.3.2}$$

For a fixed n, only a finite number of sheaves $\mathcal{D}^{ch;p}_{\Omega^{\cdot}_X;n}$ are nonzero.

If the ground ring k is a field of characteristic 0 then the sheaves $\mathcal{D}^{ch;p}_{\Omega^{\cdot}_X;n}$ and $\mathcal{D}^{ch;n-p}_{\Omega^{\cdot}_X;n}$ are in a certain sense dual to each other, see [MS].

7.4. Starting from this point we assume that $k \supset \mathbb{Q}$. According to a (superversion of) the PBW theorem, [GII], Theorem 9.18, the sheaf $\mathcal{D}^{ch}_{\Omega^{\cdot}_X}$ admits a canonical filtration such that the associated graded sheaf is canonically isomorphic to

$$gr(\mathcal{D}^{ch}_{\Omega^{\cdot}_X}) = Sym_{\Omega^{\cdot}_X}\left\{\left(\oplus_{n\geq 1}\ \Theta_{\Omega^{\cdot}_X(n)}\right) \oplus \left(\oplus_{n\geq 1}\ \Omega^1_{\Omega^{\cdot}_X(n)}\right)\right\} \tag{7.4.1}$$

Here $\Theta_{\Omega^{\cdot}_X}$ (resp. $\Omega^1_{\Omega^{\cdot}_X}$) denotes the tangent (resp. the cotangent) bundle of the supervariety $(X, \Omega^{\cdot}_X)$, and $(?)_{(n)}$ means that this bundle is put into the conformal weight n.

The endomorphisms L_0 and J_0 respect the canonical filtration; hence we get a canonical finite filtration $F.$ on each homogeneous component $\mathcal{D}^{ch;p}_{\Omega^{\cdot}_X;n}$. The graded quotients $F_i\mathcal{D}^{ch;p}_{\Omega^{\cdot}_X;n}/F_{i+1}\mathcal{D}^{ch;p}_{\Omega^{\cdot}_X;n}$ are locally free $\mathcal{O}_X$-modules of finite rank (we shall see this in the course of computations below). This allows us to introduce the elements of the Grothendieck group $K(X)$ of vector bundles

$$[\mathcal{D}^{ch;p}_{\Omega^{\cdot}_X;n}] := \sum_i\ [F_i\mathcal{D}^{ch;p}_{\Omega^{\cdot}_X;n}/F_{i+1}\mathcal{D}^{ch;p}_{\Omega^{\cdot}_X;n}] \in K(X) \tag{7.4.2}$$

Here $[E]$ in the right hand side denotes the class of a vector bundle E in $K(X)$. Consider the generating function

$$cl(\mathcal{D}^{ch}_{\Omega^{\cdot}_X})(y,q) := \sum_{p,n}\ [\mathcal{D}^{ch;p}_{\Omega^{\cdot}_X;n}]y^p q^n \in K(X)[y, y^{-1}][[q]] \tag{7.4.3}$$

7.5. For a vector bundle E over X and an indeterminate x we introduce the notations

$$[S_x E] = \sum_{i=0}^{\infty}\ [Sym^i_{\mathcal{O}_X} E] \in K(X)[[x]] \tag{7.5.1}$$

and

$$[\Lambda_x E] = \sum_{i=0}^{\infty} [\Lambda^i_{\mathcal{O}_X} E] \in K(X)[x] \tag{7.5.2}$$

The following fact was noticed in [BL] (cf. also [W]).

7.6. Theorem (L. Borisov - A. Libgober) *We have*

$$cl(\mathcal{D}^{ch}_{\Omega^{\cdot}_X})(y,q) = [\Lambda_y \Omega^1_X] \cdot \prod_{n=1}^{\infty} \big\{ [S_{q^n}\Theta_X] \cdot [S_{q^n}\Omega^1_X] \cdot [\Lambda_{y^{-1}q^n}\Theta_X] \cdot [\Lambda_{yq^n}\Omega^1_X] \big\} \tag{7.6.1}$$

7.7. *Proof.* Let us understand the bundles $\Theta_{\Omega^{\cdot}_X}$ and $\Omega^1_{\Omega^{\cdot}_X}$ a little bit more attentively.

Let us consider the local situation 3.1, with $E = \Omega$, so that $\Lambda E = \Omega^{\cdot}$. All our frames $\mathfrak{g}$ will be natural. Let $T_\psi \subset T_{\Omega^{\cdot}}$ be the A-submodule with the base $\{\psi_i\}$. The coordinate change formula (3.2.3) shows that it is a well defined A-submodule of T independent on the choice of a frame, canonically isomorphic to T.

We set

$$T_{\psi\Omega^{\cdot}} = \Omega^{\cdot} \otimes_A T_\psi \subset T_{\Omega^{\cdot}}$$

We denote by $T_{\tau\Omega^{\cdot}}$ the quotient $\Omega^{\cdot}$-module $T_{\Omega^{\cdot}}/T_{\psi\Omega^{\cdot}}$. Let $T_\tau \subset T_{\tau\Omega^{\cdot}}$ be the A-submodule generated by all τ_i. The formula (3.2.1) shows that T_τ is a well defined A-module canonically isomorphic to T, and we have

$$T_{\tau\Omega^{\cdot}} = \Omega^{\cdot} \otimes_A T_\tau$$

Returning to our variety X, we see that we get two vector bundles Θ_ψ and Θ_τ both isomorphic to Θ_X and a canonical short exact sequence

$$0 \longrightarrow \Theta_{\psi\Omega^{\cdot}_X} \longrightarrow \Theta_{\Omega^{\cdot}_X} \longrightarrow \Theta_{\tau\Omega^{\cdot}_X} \longrightarrow 0 \tag{7.7.1}$$

with

$$\Theta_{\psi\Omega^{\cdot}_X} = \Omega^{\cdot}_X \otimes_{\mathcal{O}_X} \Theta_\psi; \quad \Theta_{\tau\Omega^{\cdot}_X} = \Omega^{\cdot}_X \otimes_{\mathcal{O}_X} \Theta_\tau \tag{7.7.2}$$

Note that Θ_ψ has fermionic charge -1 and Θ_τ has fermionic charge 0.

Dually, we have two vector bundles Ω^1_ρ and Ω^1_ω both isomorphic to Ω^1_X and a canonical short exact sequence

$$0 \longrightarrow \Omega^1_{\omega\Omega^{\cdot}_X} \longrightarrow \Omega^1_{\Omega^{\cdot}_X} \longrightarrow \Omega^1_{\rho\Omega^{\cdot}_X} \longrightarrow 0 \tag{7.7.3}$$

with

$$\Omega^1_{\rho\Omega^{\cdot}_X} = \Omega^{\cdot}_X \otimes_{\mathcal{O}_X} \Omega^1_\rho; \quad \Omega^1_{\omega\Omega^{\cdot}_X} = \Omega^{\cdot}_X \otimes_{\mathcal{O}_X} \Omega^1_\omega \tag{7.7.4}$$

The bundles Ω^1_ρ, Ω^1_ω have fermionic charges 1, 0 respectively.

Note that if E is a vector bundle then

$$Sym_{\Omega^{\cdot}_X}(\Omega^{\cdot}_X \otimes_{\mathcal{O}_X} E) = \Omega^{\cdot}_X \otimes_{\mathcal{O}_X} Sym_{\mathcal{O}_X} E \tag{7.7.5}$$

Returning to PBW formula (7.4.1) we see that these remarks imply (7.6.1). △

7.8. Starting from this point let us assume that $k = \mathbb{C}$. Consider the formal power series

$$\theta(y,q) = i^{-1}(y^{1/2} - y^{-1/2})q^{1/8}\prod_{n=1}^{\infty}\{(1-q^n)(1-yq^n)(1-y^{-1}q^n)\} \tag{7.8.1}$$

It is nothing but the theta function $\theta_1(h,z)$ as defined in [HC], II, 2, §10, formula (3), p. 204, with $q = h^2$ and $y = z^2$.

If $f \in GL(V)$ is an automorphism of a d-dimendional vector space V with eigenvalues $\lambda_1, \ldots, \lambda_d$, we shall denote by $\theta_f(y,q)$ the power series

$$\theta_f(y,q) = \frac{\prod_{i=1}^d \ \theta(\lambda_i y, q)}{\prod_{i=1}^d \ \theta(\lambda_i, q)} \tag{7.8.2}$$

Let X be a proper smooth d-dimensional algebraic variety; let $g: \ X \longrightarrow X$ be a *simple* automorphism, which means by definition that the graph $\Gamma_g \subset X \times X$ is transversal to the diagonal. This implies that the set X^g if its fixed points is finite.

For each $x \in X^g$ denote by g_x the induced endomorphism of the cotangent space $\Omega^1_{X;x}$. All eigenvalues of g_x are distinct from 1.

7.9. Theorem. *Consider the power series*

$$T_{X;g}(y,q) := y^{-d/2}\sum_{a,b,n}(-1)^{a+b}Tr(g; H^a(X; \mathcal{D}^{ch;b}_{\Omega_X;n}))y^b q^n \tag{7.9.1}$$

We have

$$T_{X;g}(y,q) = \sum_{x\in X^g} \theta_{g_x}(y,q) \tag{7.9.2}$$

7.10. *Proof.* Recall that according to the Atiyah-Bott holomorphic Lefschetz fixed point formula, if E is a g-equivariant vector bundle over X then

$$\sum_i (-1)^i Tr(g; H^i(X;E)) = \sum_{x\in X^g}\frac{Tr(g;E_x)}{det(1-g_x)} \tag{7.10.1}$$

see [AB], Theorem 4.12. Note that if (V,f) are as in 7.8 then

$$Tr(g; Sym_x(V)) = \prod_{i=1}^d (1-\lambda_i x)^{-1} = Tr(g; \Lambda_{-x}(V))^{-1} \tag{7.10.2}$$

The proof of 7.6 shows that each sheaf $\mathcal{D}^{ch;p}_{\Omega_X;n}$ carries a canonical filtration whose quotients are vector bundles and the associated graded sheaf is given by the formula (7.6.1). Therefore we may apply the Lefschetz formula (7.10.1).

Note that since in the expression (7.9.1) the fermionic charge $a+b$ is taken into account, we should apply the Lefschetz formula to the element $cl(\mathcal{D}^{ch}_{\Omega_X})(-y,q)$. Due to (7.10.2) each fixed point x gives a contribution

$$y^{-d/2}\prod_{i=1}^d\left\{\frac{1-\lambda_i y}{1-\lambda_i}\prod_{n=1}^{\infty}\frac{(1-\lambda_i yq^n)(1-\lambda_i^{-1}y^{-1}q^n)}{(1-\lambda_i q^n)(1-\lambda_i^{-1}q^n)}\right\} = \theta_{g_x}(y,q)$$

where λ_i are the eigenvalues of g_x. This implies the theorem. $\triangle$.

The reader may wish to compare (7.9.2) with the explicit formulas for the trace of certain automorphisms of the Frenkel-Lepowsky-Meurman Monster vertex algebra, cf. [FLM].

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